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1991 J. Phys.: Condens. Matter 3 4213

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The quasiparticle lifetime at the mobility edge

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Received 10 September 1990, in final form 28 March 1991

Abstract. We calculate the temperature dependence of the inelastic scattering rate at the mobility edge in a disordered system in which diffusion is characterized by a wavevector and frequency dependence, and compare it with that obtained when only conventional diffusion occurs. We find that the inelastic scattering rate exhibits a linear temperature dependence in both cases. We discuss this result in the context of low-temperature transport measurements in integral quantum Hall samples, and point out certain theoretical inconsistencies which need to be resolved.

1. Introduction

The extended eigenstates of a disordered system are characterized by their ability to carry current [1,2], by which we mean that a wavepacket made up of such extended eigenstates, which is initially concentrated at some origin, will spread out and will diffuse away eventually because of impurity scattering [3]. Until recently [4], it was assumed that such diffusive motion is simple in the sense that a (diffusion) constant which is independent of wavevector and frequency appears in the diffusion equation.

In this paper we discuss diffusion which is characterized not by a constant but by a diffusion coefficient which has the following dependence on wavevector and frequency:

$$D(\mathbf{q}, \omega) = \begin{cases} D_0 & \text{for } q^2/\omega < x_0 \\ D_0(x_0\omega/q^2)^{\eta/2} & \text{for } q^2/\omega > x_0. \end{cases} \quad (1.1)$$

Diffusion characterized by the particular wavevector and frequency dependence above emerges as a general consequence of the scaling theory for localization [4], and has in fact been explicitly discussed for the integral quantum Hall effect [5]. We will henceforth refer to wavevector- and frequency-dependent diffusion as *anomalous* diffusion to distinguish it from *simple* conventional diffusion which is governed by a constant. In (1.1), the exponent η , where $\eta > 0$, controls the wavevector- and frequency-dependent component of diffusion [6]. We note that one regains simple diffusion if $\eta = 0$ in (1.1). The quantity x_0 which separates the two regimes of diffusive behaviour is found [5] to be proportional to the density of states at the Fermi energy $\rho(E_F)$, with the constant of proportionality being of order unity. It is now understood [7] that one would expect anomalous diffusion to occur generally at the mobility edge in a disordered system.

Our aim in this paper is to ask whether such wavevector- and frequency-dependent diffusion could, through its consequences on the inelastic scattering rate, be detected

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experimentally. It has been suggested [5] that a signature of anomalous diffusion, namely the non-zero parameter η , should appear in the inelastic scattering rate and hence in low-temperature transport measurements, and our aim here is to examine that conjecture. In this paper, we examine a quantity which is essentially the reciprocal of the inelastic scattering rate, namely the inelastic scattering time $\tau_{\text{in}}(E, T)$, defined as the mean time between inelastic collisions for a quasiparticle of energy E in a system at temperature T . More precisely, we calculate the temperature dependence of the inelastic scattering time for a quasiparticle at the mobility edge in a two-dimensional (2D) disordered system, in which diffusion is characterized by the wavevector and frequency dependence given in (1.1).

From a theoretical viewpoint, we can motivate our work in the following way. Because of its central role in characterizing the transport properties of physical systems, the quantity $\tau_{\text{in}}(E_{\text{F}}, T)$, which may be taken to be the lifetime of a quasiparticle at the Fermi energy, has been calculated for a variety of Fermi systems. Perhaps the best known example is the case of electrons in pure metals, where phase space arguments [8] reveal that electron–electron scattering lead to a quasiparticle damping rate proportional to T^2 .

The qualitatively simple result above is modified when impurities are introduced into the electron gas, for the following reasons. One can regard the quasiparticle damping rate as being essentially proportional to a matrix element for scattering between electron states at different energies. Such a matrix element should, in turn, depend on the strength of the scattering potential, in this case the dynamically screened Coulomb interaction, as well as on the probability densities of the corresponding electron states. When impurities are introduced into the electron gas, they cause electrons to scatter elastically, leading to diffusion on length scales greater than the mean free path, which in the simplest case is controlled by a constant that does not depend on wavevector and frequency. Hence electronic mobility in disordered systems is limited by diffusion, and the electrons spend a longer time within an arbitrary region of space than do free plane-wave electrons in pure systems. Consequently, their probability densities as well as their interactions are enhanced, thus increasing the amplitude of their matrix element for scattering. The resultant inelastic quasielectron scattering rate is then proportional to $T^{3/2}$ in an impure 3D metal [9], and proportional to T in the corresponding 2D case [10, 11].

The simple diffusive dynamics above can be described much more precisely: diffusion implies particular spatial correlations between electronic eigenstates that lie close together in energy. However, in the critical regime near the mobility edge, it is known that correlations between the eigenstates cannot be assumed to result from simple diffusion. Instead, it is widely believed that correlations in the critical regime obey a particular scaling behaviour predicted by Wegner [4]. There is a natural interpretation for these critical dynamics, namely to also associate them with a diffusion process, but with the diffusion now being wavevector- and frequency-dependent. It is interesting to ask how critical dynamics affect the quasiparticle lifetime at the mobility edge, and Belitz and Wysokinski [12] have examined this in a 3D system, where they find an inelastic scattering rate proportional to T .

In this paper we discuss an example of critical diffusive electron dynamics at the mobility edge which also obey Wegner scaling, but in which the wavevector and frequency dependence is controlled by a new scaling parameter η [5, 6]. From a theoretical viewpoint, we wish to compare the temperature dependence of the resultant inelastic electron–electron scattering rate in a 2D system with that derived from simple diffusive

dynamics, which the arguments of the previous paragraph suggest will be significantly altered.

The rest of this paper is set out as follows. In section 2 we quantitatively introduce the wavevector- and frequency-dependent diffusion coefficient $D(\mathbf{q}, \omega)$ which forms the basis of our discussion. We then investigate the implications of such anomalous diffusion for the dynamically screened Coulomb interaction using a time-dependent Lindhardt approach. We start with the exact eigenstates of the disordered system and ask what happens to these states when we turn on the dynamically screened Coulomb interaction adiabatically. We assume that we may treat this interaction to first order in perturbation theory. In this way we calculate the dynamically screened Coulomb interaction self-consistently. As anticipated, we find new dynamical screening behaviour in the presence of wavevector- and frequency-dependent diffusion.

In section 3, we calculate the quasiparticle lifetime associated with anomalous diffusion. Our derivation of the expression for the inelastic lifetime, which follows an approach due to Abrahams *et al* [10], begins with the exact eigenstates of the disordered system. We add a particle to the m th state and ask about its time evolution when the dynamically screened interaction, calculated in section 2, is turned on. We describe the effects of the interaction via the self-energy of a single-particle Green function, which we evaluate using standard diagrammatic techniques. We find that the resultant inelastic scattering rate is proportional to T . Our result is valid for disordered 2D electron systems in general, provided the screening length in the system is sufficiently small. In the special case of simple diffusion ($\eta = 0$), our result agrees with previous calculations. In addition, we find that the temperature dependence of the inelastic scattering rate is unchanged in the presence of wavevector- and frequency-dependent diffusion. This result can be understood qualitatively in terms of the matrix element for scattering discussed earlier. Wavevector- and frequency-dependent diffusion leads to changes in the probability densities of the electronic eigenstates, in other words to changes in the correlations between these eigenfunctions. Such changes inevitably also affect the strength of the dynamically screened Coulomb interaction, but the two changes in the matrix element are such that they compensate each other, leaving the temperature dependence of the scattering rate essentially unaltered.

An application of our result lies in the quantum Hall regime where the anomalous diffusion which we consider has been explicitly shown to occur. In the concluding section, we briefly consider our results in the context of experiments [13] performed on an integral quantum Hall sample, and point out some apparent contradictions which need to be resolved.

2. The response of an interacting Fermi system

Mathematically, wavevector- and frequency-dependent diffusion implies particular correlations between the eigenstates of the system. Our purpose in this section is twofold: first, to introduce these correlations, and second, to derive the dynamically screened Coulomb interaction associated with them.

It is perhaps easiest to start by considering how a wavepacket made up of the single-particle eigenstates of a disordered electronic system, spreads. For the sake of clarity as well as completeness, we repeat here arguments which have been used by other authors [3, 5, 14].

Suppose we have a wavepacket made up of extended states, which at $t = 0$ is concentrated at the origin. Such a wavepacket has a probability amplitude at point \mathbf{r} of

$$\Psi(\mathbf{r}, t = 0) = \sum_{\alpha} c_{\alpha} u_{\alpha}(\mathbf{r}) \quad (2.1a)$$

where $u_{\alpha}(\mathbf{r})$ is the eigenstate which has eigenvalue E_{α} associated with it. One finds from the requirements of completeness a corresponding coefficient given by

$$c_{\alpha} = u_{\alpha}^{*}(0) \quad (2.1b)$$

Hence after time t has elapsed, the amplitude of the wavepacket may be written

$$\Psi(\mathbf{r}, t) = \sum_{\alpha} u_{\alpha}^{*}(0) u_{\alpha}(\mathbf{r}) e^{-iE_{\alpha}t} \quad (2.2a)$$

which results in a probability density of

$$\rho(\mathbf{r}, t) \equiv |\Psi(\mathbf{r}, t)|^2 = \sum_{\alpha, \beta} u_{\beta}^{*}(\mathbf{r}) u_{\alpha}^{*}(0) u_{\alpha}(\mathbf{r}) u_{\beta}(0) e^{i(E_{\beta} - E_{\alpha})t} \quad (2.2b)$$

for the particular impurity configuration being considered.

We take other realizations of the disorder into account by averaging over all possible impurity configurations. We then obtain an averaged probability density of

$$P(\mathbf{r}, t) \equiv \langle |\Psi(\mathbf{r}, t)|^2 \rangle = \int dE \int d\omega S(\mathbf{r}, E, \omega) e^{i\omega t} \quad (2.3a)$$

in which $\langle \rangle$ denotes an impurity average,

$$\omega = E_{\beta} - E_{\alpha} \quad (2.3b)$$

and where we have identified the two-particle correlation function

$$S(\mathbf{r}, E, \omega) = \left\langle \sum_{\alpha, \beta} \delta(E - E_{\alpha} - \frac{1}{2}\omega) \delta(E - E_{\beta} + \frac{1}{2}\omega) u_{\beta}^{*}(\mathbf{r}) u_{\alpha}^{*}(0) u_{\alpha}(\mathbf{r}) u_{\beta}(0) \right\rangle. \quad (2.3c)$$

The function $S(\mathbf{r}, E, \omega)$ describes the spatial correlations between eigenstates which are separated in energy. Its Fourier transform is

$$S(\mathbf{q}, E, \omega) = \int d\mathbf{r} e^{i\mathbf{q} \cdot \mathbf{r}} S(\mathbf{r}, E, \omega). \quad (2.4)$$

This correlation function $S(\mathbf{q}, E, \omega)$, or more precisely its wavevector- and frequency-dependent component $S_0(\mathbf{q}, \omega)$, where [5]

$$S(\mathbf{q}, E, \omega) = \frac{\rho(E)}{\pi} S_0(\mathbf{q}, \omega) \quad (2.5)$$

will be a central quantity in our work.

If the initial wavepacket consists of extended states, it is natural to assume that the wavepacket spreads diffusively. The impurity-averaged probability density $P(\mathbf{r}, t)$ in (2.3a) then obeys the diffusion equation, and one may consequently show that the spectral function in (2.5) has the well-known form in 2D [3, 5, 14] of

$$S_D(\mathbf{q}, \omega) = \frac{D_0 q^2}{\omega^2 + (D_0 q^2)^2} \quad (2.6)$$

where D_0 is the usual diffusion constant that is independent of wavevector and frequency. The divergence of $S_D(\mathbf{q}, \omega)$ in the limit $\omega \rightarrow 0, q \rightarrow 0$ is a signature of the large fluctuations in the local density of states which result from diffusion [14].

However, it now appears that correlations between the extended eigenstates of a disordered system are not as simple as one might expect from the above arguments. In fact, in an integral quantum Hall system [5], wavevector- and frequency-dependent correlations of the form

$$S_A(\mathbf{q}, \omega) = \frac{D_0 q^2}{\omega^2 + (D_0 q^2)^2} \quad (2.7a)$$

for $q^2/\omega < x_0$, and

$$S_A(\mathbf{q}, \omega) = \frac{D_0 (x_0 \omega / q^2)^{\eta/2} q^2}{\omega^2 + [D_0 (x_0 \omega / q^2)^{\eta/2} q^2]^2} \quad (2.7b)$$

for $q^2/\omega > x_0$, have been found, where $x_0 = c\rho(E_F)$, $\rho(E_F)$ being the density of states at the Fermi energy and c being a number of order unity.

These correlations can be understood in the following way. For short wavevectors $\xi^{-1} < q < L_0^{-1}$ where ξ is the localization length and $L_0 = (\omega x_0)^{-1/2}$ is a critical length which depends on frequency, correlations between the states are characterized by a diffusion constant D_0 that is independent of wavevector and frequency. These are what we have called simple correlations. For longer wavevectors $L_0^{-1} < q < l_{\text{elastic}}^{-1}$ correlations are characterized not by a diffusion constant but by the wavevector- and frequency-dependent diffusion coefficient of (1.1b). This we have called anomalous diffusion.

In real space, (2.7) imply that on long length scales $L_0 < l < \xi$, the wavepacket diffuses in the conventional way, but its motion at intermediate length scales $l_{\text{elastic}} < l < L_0$ appears to be controlled by a new parameter η . In addition, the wavepacket is expected to move ballistically on short length scales $l < l_{\text{elastic}}$.

In the rest of this paper, we will work with dimensionless parameters wherever possible. The product $D_0 x_0$ forms one dimensionless quantity whose value may be shown to be of order unity. We can set the value of $D_0 x_0$ equal to unity with no important changes to the physics. This then allows us to characterize the two regimes of behaviour in (1.1) and (2.7) rather simply: a second dimensionless parameter, namely $D_0 q^2/\omega$, drops out quite naturally, and simple diffusion is obtained for $D_0 q^2/\omega < 1$, whilst wavevector- and frequency-dependent diffusion occurs for $D_0 q^2/\omega > 1$.

In the rest of this section, we will examine the consequences of the anomalous correlations above on screening in the system. It is obvious from the way in which we have introduced the function $S_0(\mathbf{q}, \omega)$ that these correlations are directly connected to elastic relaxation processes in the disordered electronic system. We would therefore

qualitatively expect such correlations to be effective in determining the dynamical response of the system to an applied potential. A potential which enters quite naturally into the problem is that due to the Coulomb interaction between the electrons. The response of the system to this potential, in the form of the dynamically screened Coulomb interaction, should therefore be related to the correlations $S_0(\mathbf{q}, \omega)$. We now discuss that connection in detail.

The relationship between the dynamically screened Coulomb interaction $V_s(\mathbf{q}, \omega)$ and the bare Coulomb potential $V_b(\mathbf{q})$ is given by [15]

$$V_s(\mathbf{q}, \omega) = \frac{V_b(\mathbf{q})}{1 + V_b(\mathbf{q})\chi(\mathbf{q}, \omega)} \quad (2.8)$$

where $\chi(\mathbf{q}, \omega)$ is the dynamical susceptibility. The dynamical susceptibility may be regarded as the response of a system to an applied potential, on the microscopic level. In our case the dynamically screened Coulomb interaction provides an applied potential, and the electrons in the system respond by rearranging themselves, thereby providing an induced charge density ρ^1 . Hence we have as the definition of susceptibility

$$\chi(\mathbf{q}, \omega) = \frac{\rho^1(\mathbf{q}, \omega)}{V_s(\mathbf{q}, \omega)} \quad (2.9)$$

We first calculate the susceptibility by assuming a dynamically screened interaction which is perturbatively weak. We then substitute our result for the susceptibility into (2.8) to determine the dynamically screened Coulomb interaction self-consistently.

The problem of evaluating the dynamical susceptibility defined in (2.8) is essentially one of calculating the charge density induced by a time-dependent applied potential. The following approach, based on time-dependent perturbation theory, is due to Lindhardt [16].

We again take as our starting point the set of single-electron disordered eigenstates $u_\alpha(\mathbf{r})$, which have corresponding energy levels E_α . In a non-interacting system at $T = 0$ we imagine these levels to be filled in accordance with the Pauli principle, up to the Fermi energy E_F .

We then imagine turning on the dynamically screened Coulomb interaction $V_s(\mathbf{r}, t)$ adiabatically. We assume this interaction to be a first-order perturbation on the disordered eigenstates. The effect of the perturbation is to cause transitions between the single-electron eigenstates. Hence the solution of the full Hamiltonian can be written as a linear superposition

$$\psi(\mathbf{r}, t) = \sum_\alpha c_\alpha(t) u_\alpha(\mathbf{r}) e^{-iE_\alpha t} \quad (2.10)$$

whose time-dependent coefficients $c_\alpha(t)$ are given by perturbation theory. We calculate the charge density $|\psi(\mathbf{r}, t)|^2$ induced by the dynamically screened Coulomb interaction, average this over all possible impurity configurations, and take the appropriate Fourier transforms to obtain the dynamical susceptibility. This leads us to the following general expression for the dynamical response:

$$\chi(\mathbf{q}, \omega) = \frac{s\rho(E_F)}{\pi} \int_{-\infty}^{\infty} d\omega' \frac{\omega'}{\omega' - \omega} S(\mathbf{q}, \omega') + is\rho(E_F)\omega S(\mathbf{q}, \omega). \quad (2.11)$$

In (2.11), s is the spin degeneracy and ρ denotes the principal value of the integral. This equation tells us precisely how the dynamical response of a system is connected with the correlations between its single-particle eigenstates. Once these correlations are known, the imaginary part of the susceptibility follows through trivially, but its real component requires the evaluation of a frequency integral.

Omitting details (see appendix 1), we find that if diffusion in the system is simple, the dynamical susceptibility is given by

$$\chi_D(\mathbf{q}, \omega) = 2\rho(E_F) \left(\frac{D_0 q^2}{\omega^2 + (D_0 q^2)^2} \right) (D_0 q^2 + i\omega) \tag{2.12}$$

which agrees with a diagrammatic calculation [15, 17].

The resultant dynamically screened Coulomb interaction determined self-consistently for these simple correlations is

$$V_s(\mathbf{q}, \omega) = \left(\frac{2\pi e^2}{q} \right) \frac{D_0 q^2 - i\omega}{D_0 \kappa q + D_0 q^2 - i\omega} \tag{2.13}$$

where κ is the inverse screening length, equal to $4\pi e^2 \rho(E_F)$.

If the correlations are anomalous and $\eta \neq 0$, we find a susceptibility of

$$\chi_A(\mathbf{q}, \omega) = \rho(E_F) \left(\frac{D_0 q^2}{\omega^2 + (D_0 q^2)^2} \right) (D_0 q^2 + 2i\omega) \tag{2.14a}$$

for $|\omega| > D_0 q^2$, and

$$\chi_A(\mathbf{q}, \omega) = \rho(E_F) \frac{g(\eta)(D_0 q^2)^2}{\omega^2 + (D_0 q^2)^2} + 2i\rho(E_F) \frac{\omega(\omega/D_0 q^2)^{\eta/2} D_0 q^2}{\omega^2 + (D_0 q^2)^2 (\omega/D_0 q^2)^\eta} \tag{2.14b}$$

for $|\omega| < D_0 q^2$, where $g(\eta)$ is an undetermined function of η of $O(\pi)$. When $\eta = 0$, $g(\eta) = 1$. This response clearly suggests the presence of new dynamical screening behaviour (cf (2.12)).

One could obviously write down a self-consistent expression for the dynamically screened Coulomb interaction by substituting (2.14) into (2.8), but the result is far too unwieldy to be useful. Instead, we analyse the resulting expression in different wavevector and frequency regimes and determine the approximate form of the dynamically screened Coulomb interaction in each region.

When $D_0 q^2 < \omega$, diffusion is simple, and we find a dynamically screened Coulomb interaction much the same as before:

$$V_s(\mathbf{q}, \omega)]_D = \frac{2\pi e^2}{q} \frac{D_0 q^2 - i\omega}{D_0 q^2 - i\omega + D_0 \kappa q/2} \tag{2.15}$$

The above screened interaction is therefore approximately

$$V_s(\mathbf{q}, \omega)]_D \simeq \frac{2\pi e^2}{q} \tag{2.16a}$$

for $q > \kappa$, and

$$V_s(\mathbf{q}, \omega)]_D = \left(\frac{2\pi e^2}{q} \right) \frac{D_0 q^2 - i\omega}{D_0 \kappa q/2 - i\omega} \tag{2.16b}$$

for $q \leq \kappa$.

On the other hand, when $D_0 q^2 > \omega$, we find the following approximate form for the dynamically screened Coulomb interaction:

$$V_s(q, \omega)]_A \simeq \frac{2\pi e^2}{q} \quad (2.17a)$$

for $q > \kappa$, and for $q < \kappa$

$$V_s(q, \omega)]_A \simeq \frac{2\pi e^2}{q} \left(\frac{(D_0 q^2)^2 + \omega^2}{\omega^4 (D_0 \kappa q / \omega)} \right) \left(\frac{(\tilde{D}(q, \omega) q^2)^2 + \omega^2}{(D_0 q^2 / \omega)^{6-2\eta} + (D_0 q^2 / \omega)^{4-\eta}} \right) \\ \times \left(\left(\frac{D_0 q^2}{\omega} \right)^{3-\eta} - i \left(\frac{D_0 q^2}{\omega} \right)^{2-\eta/2} \right) \quad (2.17b)$$

where

$$\tilde{D}(q, \omega) = D_0 \left(\frac{x_0 \omega}{q^2} \right)^{\eta/2}$$

Our results therefore indicate that the anomalous diffusion which is characteristic at longer wavevectors does not in fact lead to any important changes in the dynamically screened Coulomb interaction in the system, if one considers wavevectors much longer than the inverse screening length (cf (2.16a) and (2.17a)). But at wavevectors much shorter than or of order of the inverse screening length, a novel form for the dynamically screened Coulomb interaction appears which is controlled by the non-zero parameter η (cf (2.17b)), and which reduces to give the more familiar screening behaviour associated with simple diffusion if $\eta = 0$. In the rest of this paper, we will explore one of the possible experimental consequences of these unusual screening dynamics.

3. Calculation of the quasiparticle lifetime

In this section we calculate the inelastic quasielectron scattering rate when anomalous diffusion is present in a system. For a quasielectron at the Fermi energy, the scattering rate should, after averaging over disorder, depend only on the temperature of the system. The simplest temperature dependence has the power-law form

$$\Gamma_{\text{in}}(E_F, T) = aT^p \quad (3.1)$$

where a and p are numbers which depend on the symmetries in a system, its dimensionality, and on what scattering mechanisms are at work. In this section we consider a disordered two-dimensional system in which wavevector- and frequency-dependent diffusion is present, and determine the associated exponent p and constant of proportionality a . We wish in particular to compare our value of p with that derived for a similar system in which only simple diffusion is present [10, 11, 15, 17].

Technically, the decay rate of a quasiparticle can be obtained by calculating its total self-energy shift and then taking the imaginary part [10]. The self-energy of a quasielectron of energy E in a disordered system is

$$\sum_E(\omega, T) = \frac{1}{\rho(E_F)} \left\langle \sum_m \delta(E - E_m) \Sigma_m(\omega, T) \right\rangle \quad (3.2)$$

where $\Sigma_m(\omega, T)$ is the self-energy shift of the electron in the m th disordered eigenstate $u_m(\mathbf{r})$. One can calculate this self-energy shift in the m th level by using standard techniques in many-body perturbation theory [18].

In evaluating the averaged self-energy, we assume that we need only consider the first-order contribution to the self-energy $\Sigma_m(\omega, T)$. This first-order shift consists of Hartree-Fock corrections to the unperturbed level: there is a Hartree term which describes the average interaction between the electron and the non-uniform background, and an exchange term from pairs of electrons with parallel spin which arises because of the Pauli exclusion principle. For a long-ranged interaction such as the dynamically screened Coulomb potential, the exchange interaction is much larger than the Hartree term [10, 19].

We therefore consider only the exchange term, whose diagrammatic representation consists of a Matsubara Green function and an interaction line [10]. This diagram can be evaluated using the Feynman rules for a many-electron system at a finite temperature, but the usual recipes must be extended when dealing with a disordered system to include non-trivial correlations between the electronic eigenstates. Abrahams *et al* [10] generalized the standard approach to cover all possible correlations. Using their method, we find (see appendix 2) that the temperature dependence of the inelastic scattering rate for a quasielectron at the Fermi energy $E = E_F = 0$ is given by

$$\tilde{\Gamma}(T) = \frac{-2}{\pi} \int_{-\infty}^{\infty} \frac{dy}{\sinh(\beta y)} \int \frac{d^2q}{(2\pi)^2} \text{Im}[V_s(\mathbf{q}, y)]S(\mathbf{q}, y). \tag{3.3}$$

The double integral in (3.3) can only be calculated analytically if we make sensible approximations for the integrand. Such simplification is necessary even in the relatively straightforward case of conventional diffusion. Using ideas outlined by Fukuyama and Abrahams [20], we can reduce the above integral (see appendix 2) in the wavevector and frequency range $D_0q^2/\omega < 1$, for which $\eta = 0$, to

$$\left. \frac{1}{\tau_{in}} \right]_{\eta=0} \simeq \frac{-2}{\pi} \int_{-k_B T}^{k_B T} \frac{dy}{\beta y} \int_0^{(\omega/D_0)^{1/2}} \frac{dq 2\pi q}{(2\pi)^2} \left(\frac{-4\pi e^2}{q} \right) \frac{D_0 \kappa q y}{(D_0 \kappa q)^2 + 4y^2} \frac{D_0 q^2}{(D_0 q^2)^2 + y^2}. \tag{3.4a}$$

The main contributions to the frequency integral in (3.4a) come from small frequencies which, however, lead to an infrared divergence. We cure this singularity by demanding that the calculation be self-consistent, so that frequencies with an amplitude below $1/\tau_{in}$ are assumed not to contribute to the determination of τ_{in} itself. In addition one can show that in the wavevector and frequency range considered, [11, 17]

$$D_0 \kappa q \gg y \tag{3.4b}$$

and this allows the integral in (3.4a) to be simplified further, to

$$\left. \frac{1}{\tau_{in}} \right]_{\eta=0} \simeq \frac{8e^2 D_0 \kappa k_B T}{\pi} \int_{1/\tau_{in}}^{k_B T} \frac{dy}{y} \int_{y/D_0 \kappa}^{(\omega/D_0)^{1/2}} \frac{dq q}{(D_0 \kappa q)^2 y^2 + (D_0 q^2)^2} \tag{3.4c}$$

which gives a scattering rate of

$$\left. \frac{1}{\tau_{in}} \right]_{\eta=0} \simeq \frac{e^2 k_B T}{D_0 \kappa} \ln \left(\frac{D_0 \kappa}{2e^2} \right). \tag{3.5}$$

A similar analysis for the $\eta \neq 0$ component (appendix 2) gives us an inelastic scattering rate of

$$\left. \frac{1}{\tau_{\text{in}}} \right]_{\eta \neq 0} \simeq \frac{2e^2 F(\eta)}{\pi D_0 \kappa} k_B T \ln \left(\frac{\pi D_0 \kappa}{2e^2 F(\eta)} \right) \quad (3.6a)$$

in a disordered system with anomalous diffusion, where

$$F(\eta) = \int_1^{\infty} \frac{dx(1+x^2)}{x^2(1+x^2-\eta)}. \quad (3.6b)$$

Equations (3.6) indicate that anomalous correlations produce an inelastic scattering rate which varies linearly with temperature, with the parameter η appearing only in the constant of proportionality. This temperature dependence is the same as that due to purely conventional diffusion. We have therefore shown that, perhaps a little surprisingly, anomalous diffusion does not in fact change the inelastic scattering rate in the system.

We can understand this result qualitatively by regarding the scattering rate as being essentially proportional to a matrix element for scattering between electron states at different energies. Such a matrix element in turn depends on the strength of the dynamically screened Coulomb interaction as well as on correlations between the electronic eigenfunctions. Changes in the eigenstate correlations also affect the strength of the dynamically screened interaction, and the two changes are such that they compensate one another, leaving the temperature dependence of the scattering rate essentially unchanged.

4. A comparison between theory and experiment: some results on the integral quantum Hall effect

A strong motivation for examining the anomalous diffusion discussed in this paper lies in the fact that such diffusive dynamics have been explicitly found in an integral quantum Hall system. In this section, we consider some low temperature transport measurements which Wei *et al* [13] have carried out on a semiconductor heterostructure, and discuss our results in the context of these experiments.

A dimensionless parameter which enters naturally into a discussion of low-temperature electronic transport properties is the ratio $\beta = p/2\nu$, where p is the inelastic scattering exponent already introduced, and ν is the critical index which governs the divergence of the localization length. We will not enter into a discussion of the relationship between p and ν here, but instead refer the reader to [21].

In an integral quantum Hall system, simple physical arguments indicate that both the maxima of the slopes of the Hall steps, as well as the reciprocals of the corresponding ranges of non-zero dissipative conductivity, should scale as a function of temperature with the ratio β . The low-temperature conductivity measurements performed by Wei *et al* [13] on a gallium arsenide sample confirm this scaling hypothesis, and provide an experimental value of 0.42 ± 0.04 for the ratio β . Wei *et al* interpret this result in the following way: they assume that $p = 1$ and conclude therefore that their measured value for β is consistent with a localization exponent $\nu \simeq 1.2$.

Unfortunately, the picture as we see it does not appear to be so clear-cut. We agree that current evidence suggests that ν should have a value $\simeq 1$ in a disordered system which does not have a magnetic field imposed on it, at least in 3D. However, there has been increasing evidence in recent years that in quantum Hall systems, and in other disordered systems without time-reversal symmetry, the value of the localization exponent ν is actually closer to 2. This empirical observation is based on the best available theoretical estimates for ν in such systems, calculated using a variety of analytical as well as numerical techniques. The following values of ν have been obtained theoretically: $\nu = 1.9 \pm 0.2$ (from an analytical perturbational calculation, [22]); $\nu \simeq 2$ (an estimate from numerical calculations in [23]); $\nu = 2.5 \pm 0.5$ (based on numerical results, [24]); $\nu = 7/3$ (from an analytical, semiclassical calculation in [25]); and $\nu = 2.34 \pm 0.04$ (from a numerical calculation, [26]).

The above results appear consistent with the value of the localization exponent ν being approximately 2 in disordered systems with broken time-reversal symmetry. Taking ν to be equal to 2, we find from the result of Wei *et al* that for β that $p \simeq 1.7$. However, our calculation in this chapter indicates that p should be unity if we assume electron-electron scattering to be the dominant inelastic scattering mechanism in the system.

In order to reconcile these apparently inconsistent results, one is forced to one of two possible conclusions. The first possibility is that electron-electron scattering is not the dominant inelastic scattering process, in which case p need not be unity, and the above results are consistent with one another. Alternatively, exactly the opposite could be true, namely that Coulomb interactions in quantum Hall and disordered systems generally are actually much stronger than has been allowed for, by which we mean that Coulomb effects should be taken into account even when determining properties associated with localization, for instance the value of the localization exponent ν . This has not so far been done: localization has essentially been studied by treating Coulomb interactions as either weak or non-existent; in particular, the estimates of ν quoted above derive from a non-interacting single-particle picture.

Both of the above propositions have interesting physical consequences. The first implies that Coulomb interactions in quantum Hall systems are extremely well screened, in which case electron-electron scattering cannot be the dominant inelastic scattering process in these disordered systems, and one must consider an alternative scattering mechanism which dominates. The most obvious candidate is then electron-phonon scattering, and this conclusion seems a little surprising since we are dealing with temperatures as low as 0.1 K. The second proposition, if true, is equally surprising because it suggests that a proper understanding of localization must, even in principle, involve taking Coulomb interactions into account. This would imply that a radical examination of existing single-particle theories would be necessary for future progress. Further work must be done before the issues raised above can be resolved.

5. Summary

In this paper we calculate the quasiparticle scattering rate in a disordered electron system in which the electronic dynamics are characterized by a diffusion coefficient that depends on wavevector and frequency. The appearance of wavevector- and frequency-dependent diffusion necessarily implies new dynamical screening in the system. We first calculate the dynamical susceptibility, and then use our result to determine the

dynamically screened Coulomb interaction in the system self-consistently. We find that although the response of the electrons to an external perturbation is indeed rather different from what one would expect if they diffused conventionally, the temperature-dependence of the associated inelastic quasielectron scattering rate is unchanged in these circumstances, being proportional to temperature just as it is for a system in which only conventional diffusion is present.

We can understand this result in simple terms. The inelastic scattering rate is essentially proportional to the amplitude of a matrix element for scattering between electron states at different energies. Such a matrix element depends in turn on the strength of the dynamically screened Coulomb interactions, as well as on correlations between the electron eigenstates. Changes in the eigenstate probability densities also affect the strength of the dynamically screened interaction, and the two changes compensate one another, leaving the temperature dependence of the scattering rate unchanged.

On the other hand, one would expect from known theoretical and experimental results that the inelastic scattering rate in a disordered system such as the one we consider should increase approximately as a quadratic function of temperature. There are two possibilities which could explain these rather contradictory results: either electron-phonon scattering (and not Coulomb scattering) is the dominant inelastic scattering mechanism in these systems down to temperatures of ~ 0.1 K, or Coulomb interactions in disordered systems are much stronger than has hitherto been assumed.

Appendix 1.

For anomalous correlations, the frequency integral in (2.11) becomes non-trivial. Letting $D_0 q^2 = \omega_0$ and setting the dimensionless quantity $D_0 x_0$ to unity, we can rewrite the frequency integral in (2.11) as

$$\wp \int_{-\infty}^{-\omega_0} \frac{d\omega'}{\omega' - \omega} \frac{\omega_0 \omega'}{\omega'^2 + \omega_0^2} + \wp \int_{\omega_0}^{\infty} \frac{d\omega'}{\omega' - \omega} \frac{\omega_0 \omega'}{\omega'^2 + \omega_0^2} + \wp \int_{-\omega_0}^{\omega_0} \frac{d\omega'}{\omega' - \omega} \frac{\omega_0 \omega' (\omega'/\omega_0)^{\eta/2}}{\omega'^2 + \omega_0^2 (\omega'/\omega_0)^\eta}. \quad (\text{A1.1})$$

The dimensionless ratio ω_0/ω which now appears quite naturally provides us with two obvious limits, namely $\omega_0/\omega \ll 1$ and $\omega_0/\omega \gg 1$, in which each integral in (A1.1) simplifies and can be evaluated. We then estimate the real part of the dynamical susceptibility by interpolating between these limits.

The detailed analysis is rather complicated. For example, let us consider the sum of the first two integrals in (A1.1). In the limit $\omega/\omega_0 \ll 1$, all frequencies over which integration occurs are large compared with ω . Hence we can expand the first term in both integrands:

$$\frac{1}{\omega' - \omega} \simeq \frac{1}{\omega'} \left(1 + \frac{\omega}{\omega'} \right).$$

Retaining only lowest-order terms, we obtain

$$\wp \int_{-\infty}^{-\omega_0} d\omega' \frac{\omega_0}{\omega'^2 + \omega_0^2} + \wp \int_{\omega_0}^{\infty} d\omega' \frac{\omega_0}{\omega'^2 + \omega_0^2} = \frac{\pi}{2}.$$

Similarly, in the limit $\omega_0/\omega \ll 1$, the sum of these integrals is $(\pi/2)(\omega_0/\omega)^2$, to leading order. A possible function which interpolates between these limits is

$$\frac{\pi}{2} \frac{\omega_0^2}{\omega^2 + \omega_0^2}$$

We therefore estimate a contribution to the dynamical susceptibility of

$$\rho(E_F) \frac{\omega_0^2}{\omega^2 + \omega_0^2}$$

from the simple part of the anomalous correlations, which not surprisingly is of the same form as (2.12).

Using the same approach, we can estimate the contribution to the susceptibility from correlations that are characterized by $\eta \neq 0$, the final result being shown in (2.14).

Appendix 2.

Following Abrahams *et al* we obtain an averaged self-energy which is directly related to the correlation function introduced in section 2:

$$\sum_E(\omega, T) = \frac{k_B T}{\pi} \sum_{q, \omega_n} V_s(q, |\omega_n|) \int_{-\infty}^{\infty} \frac{d\nu}{\nu - i(\omega - \omega_n)} S(q, \nu - E). \tag{A2.1a}$$

Here the terms $\omega_n = 2\pi n k_B T$ where $n = 0, 1, 2, \dots$ are the Matsubara frequencies, and

$$V_s(q, |\omega_n|) = \left(\frac{2\pi e^2}{q} \right) \frac{D(q, |\omega_n|)q^2 + |\omega_n|}{D(q, |\omega_n|)\kappa q + |\omega_n|} \tag{A2.1b}$$

is the dynamically screened propagator in 2D [15]. Performing the integral over ν in (2.1a), summing over Matsubara frequencies, and taking the imaginary part of the resultant self-energy, we obtain a quasidelectron scattering rate of

$$\tilde{\Gamma}_E(E, T) = \frac{-2}{\pi} \int_{-\infty}^{\infty} dy [f(y + E) + n(y)] \int \frac{d^2 q}{2\pi^2} \text{Im}[V_s(q, y)] S(q, y). \tag{A2.2a}$$

For a quasidelectron at the Fermi energy $E = E_F = 0$, we must evaluate (cf (3.3))

$$\tilde{\Gamma}(T) = \frac{-2}{\pi} \int_{-\infty}^{\infty} \frac{dy}{\sinh(\beta y)} \int \frac{d^2 q}{(2\pi)^2} \text{Im}[V_s(q, y)] S(q, y). \tag{A2.2b}$$

We now describe how we estimate the $\eta = 0$ contribution to the inelastic scattering rate.

In this case we use (2.16) to approximate

$$\text{Im}[V_s(q, y)]_D \simeq \frac{-4\pi e^2}{q} \frac{D_0 \kappa q y}{(D_0 \kappa q)^2 + 4y^2} \tag{A2.3}$$

for $q \leq \kappa$ and zero otherwise, i.e. for all $q > \kappa$.

It is also sensible to let

$$\frac{1}{\sinh(\beta y)} \simeq \begin{cases} 1/\beta y & \text{for } y \leq k_B T \\ 0 & \text{for } y > k_B T. \end{cases} \quad (\text{A2.4})$$

because the contributions from large frequencies are small. We therefore obtain (cf (3.4))

$$\frac{1}{\tau_{\text{in}}}]_{\eta=0} \simeq \frac{-2}{\pi} \int_{-k_B T}^{k_B T} \frac{dy}{\beta y} \int_0^{(\omega/D_0)^{1/2}} \frac{dq 2\pi q}{(2\pi)^2} \left(\frac{-4\pi e^2}{q} \right) \frac{D_0 \kappa q y}{(D_0 \kappa q)^2 + 4y^2} \frac{D_0 q^2}{(D_0 q^2)^2 + y^2} \quad (\text{A2.5})$$

and this eventually leads to the result of (3.5). The contribution from the $\eta \neq 0$ component is not much more complicated. In this case, we let

$$\text{Im}V_s(\mathbf{q}, y)]_A \simeq \frac{2\pi e^2}{q} \left(\frac{(D_0 q^2)^2 + y^2}{y^4 (D_0 \kappa q / y)} \right) \left(\frac{\tilde{D}(\mathbf{q}, y) q^2 + y^2}{(D_0 q^2 / y)^{6-2\eta} + (D_0 q^2 / y)^{4-\eta}} \right) \left(\frac{D_0 q^2}{y} \right)^{2-\eta/2} \quad (\text{A2.6})$$

for $q < \kappa$, and zero for $q > \kappa$ respectively. Although this appears a rather complicated function, its product with the correlations $S_A(\mathbf{q}, \omega)$ is actually rather simple because factors in the numerator and denominator cancel out. Thus we obtain the scattering rate of (3.6).

Acknowledgments

I would like to thank John Chalker for suggesting this problem and for the many useful discussions we have had. I would also like to thank one of my referees for pointing out the work of B Huckenstein *et al* as well as that contained in [25].

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